Quantum error correction of coherent errors by randomization

O. Kern[1], G. Alber[1,a], and D.L. Shepelyansky[2]

[1] Institut für Angewandte Physik, Technische Universität Darmstadt, 64289 Darmstadt, Germany
[2] Laboratoire de Physique Théorique, UMR 5152 du CNRS, Université Paul Sabatier, 31062 Toulouse Cedex 4, France

Abstract. A general error correction method is presented which is capable of correcting coherent errors originating from static residual inter-qubit couplings in a quantum computer. It is based on a randomization of static imperfections in a many-qubit system by the repeated application of Pauli operators which change the computational basis. This Pauli-Random-Error-Correction (PAREC)-method eliminates coherent errors produced by static imperfections and increases significantly the maximum time over which realistic quantum computations can be performed reliably. Furthermore, it does not require redundancy so that all physical qubits involved can be used for logical purposes.

PACS. 03.67.Lx Quantum computation – 03.67.Pp Quantum error correction and other methods for protection against decoherence – 05.45.Mt Quantum chaos; semiclassical methods

Current developments in quantum physics demonstrate in an impressive way its technological potential [1]. In quantum computation, e.g., characteristic quantum phenomena, such as interference and entanglement, are exploited for solving computational tasks more efficiently than by classical means [2–6]. However, these quantum phenomena are affected easily by unknown residual inter-qubit couplings or by interactions with an uncontrolled environment [7]. In order to protect quantum algorithms against such undesired influences powerful methods of error correction have been developed over the last years.

So far techniques of quantum error correction have concentrated predominantly on decoherence caused by uncontrolled couplings to environments [8,9]. In these cases appropriate syndrome measurements and recovery operations can reverse errors. However, up to now much less is known about the correction of coherent, unitary errors. Even if a quantum information processor (QIP) is isolated entirely from its environment and if all quantum gates are performed perfectly, there may still be residual inter-qubit couplings affecting its performance. Recently, it was demonstrated that static imperfections, i.e. random inter-qubit couplings which remain unchanged during a quantum computation, restrict the computational capabilities of a many-qubit QIP significantly as they cause quantum chaos and quantum phase transitions [10]. Furthermore, in addition to a usual exponential decay such static imperfections also cause a Gaussian decrease of the fidelity with time. At sufficiently long times this Gaussian decrease dominates the decay of the fidelity thus limiting

significantly the maximum reliable computation times of many-qubit QIPs [11,12].

In this paper a general error correcting method is presented for overcoming these disastrous consequences of static imperfections. It is based on the repeated random application of Pauli operators to all the qubits of a QIP. The resulting random changes of the computational basis together with appropriate compensating changes of the quantum gates slow down the rapid Gaussian decay of the fidelity and change it to a linear-in-time exponential one. As a result this Pauli-Random-Error-Correction (PAREC)-method increases significantly the maximum time scale of reliable quantum computation. In addition, neither control measurements nor redundant qubits are required so that all physical qubits are logical qubits.

In order to put the problem into perspective let us concentrate on the quantum algorithm of the quantum tent-map as a particular example [12]. One iteration of this special case of a quantum-rotator-map is governed by the unitary operator

$$\hat{U} = e^{-iT\hat{p}^2/(2m\hbar)} e^{-ikV(\hat{x})/\hbar}. \tag{1}$$

It describes the one-dimensional dynamics of a periodically kicked particle of mass m. The operators $\hat{p}$ and $\hat{x}$ denote momentum and position operators and T is the period of the kicks of magnitude $kV(x)$. The dynamics of the particle is assumed to be confined to the spatial interval $0 \leq x \leq l$ with periodic boundary conditions. The name of this quantum-map originates from the force which resembles the form of a tent, i.e.

$$-V'(x) = \begin{cases} (x - l/4), & (0 \leq x < l/2) \\ (3l/4 - x), & (l/2 \leq x < l). \end{cases} \tag{2}$$

a e-mail: gernot.alber@physik.tu-darmstadt.de

Due to the periodic boundary conditions the momentum eigenvalues are given by $p_n = 2\pi\hbar n/l$ with $n \in \mathbb{Z}$. Imposing the 'resonance condition' $T = [m(l/2\pi)^2/\hbar](2\pi/N)$ with $N \in \mathbb{N}$ implies the symmetry $\langle p_{n+N}|\hat{U}|p_{n'+N}\rangle = (-1)^N \langle p_n|\hat{U}|p_{n'}\rangle$ and $\hat{U}$ decomposes into a direct sum of $N \times N$ matrices [13]. Thus, for a given value of N the dynamics of the quantum tent-map can be simulated on a quantum computer (QC) with n_q qubits provided $N = 2^{n_q}$. In this case the unitary operation of equation (1) can be performed with the help of $n_g = (9/2)n_q^2 - (11/2)n_q + 4$ universal quantum gates, i.e. Hadamard-, phase-, controlled-phase-, and controlled-not gates [12]. The classical limit of the quantum tent-map corresponds to $T \rightarrow 0$, $k \rightarrow \infty$ with $K = kT[l/(2\pi\hbar)]/[m(l/2\pi)^2/\hbar]$ remaining constant. In this parameter regime the tent-map exhibits all complex dynamical features characteristic for quantum chaos [14].

In order to model static imperfections we assume that the n_q qubits of a realistic QC are coupled by random Heisenberg-type nearest-neighbour interactions as described by the Hamiltonian

$$\hat{H} = \sum_{i=0}^{n_q-1} \delta_i \hat{Z}_i + \sum_{i=0}^{n_q-2} J_i(\hat{X}_i\hat{X}_{i+1} + \hat{Y}_i\hat{Y}_{i+1} + \hat{Z}_i\hat{Z}_{i+1}) \quad (3)$$

with the Pauli (spin) operators $\hat{X}, \hat{Y}, \hat{Z}$. The quantities δ_i and J_i denote the strengths of the detuning and the nearest-neighbour interaction of qubit i. In the case of static imperfections these quantities are distributed randomly and homogeneously in the energy-interval $[-\sqrt{3}\eta, \sqrt{3}\eta]$ and remain static in time during a quantum computation.

In general, after t iterations of a quantum map the fidelity, defined through the ideal and the perturbed quantum states $|\psi(t)\rangle$ and $|\psi_\eta(t)\rangle$, i.e. $f = |\langle\psi(t)|\psi_\eta(t)\rangle|^2$, decays according to [12]

$$-\ln f(t) = \frac{t}{t_c} + \frac{t^2}{t_c t_H}. \quad (4)$$

This relation is valid as long as η and t are sufficiently small so that the fidelity f remains close to unity. The time scale t_c governing the linear-in-time exponential decay is determined by Fermi's Golden rule. In particular, in recent simulations [12] it was found to be inversely proportional to $(n_q n_g^2 \eta^2)$. The second characteristic time scale entering equation (4) is the Heisenberg time $t_H \approx 2^{n_q}$ which is determined by the dimension of the n_q-qubit Hilbert space. According to equation (4) the linear-in-time exponential decay changes to the much more rapid quadratic Gaussian decay roughly after $t \approx t_H$ iterations. Such quadratic-in-time Gaussian decays are characteristic for coherent dephasing phenomena. The quadratic fidelity decrease of equation (4) is drastically limiting the maximum time over which a quantum computation can be performed reliably. Contrary to static imperfections, random imperfections which change from gate to gate, e.g., lead to a purely linear-in-time exponential decay of the fidelity [12]. This suggests the idea that a randomization of

static imperfections might help to slow down the time evolution of the fidelity from a quadratic-in-time exponential decay to a linear-in-time one thus increasing the reliable computation time significantly.

But how can a randomization of static imperfections be achieved efficiently? A basic idea of quantum error correction is to exploit the freedom of choice of the computational basis for an appropriate encoding. This idea can also be used for an efficient randomization of static imperfections by changing the computational basis repeatedly and randomly during a quantum computation. However, to leave the quantum algorithm unchanged these basis changes have to be compensated by appropriate transformations of the universal quantum gates.

In order to address this issue let us concentrate on the particular example of the quantum tent-map algorithm and on the static Heisenberg-type imperfections described by equation (3). A convenient way of realizing such random changes of the computational basis is to apply repeatedly randomly selected Pauli operators to all the n_q qubits of the QC. For this purpose it is advantageous to represent the elementary quantum gates of the quantum algorithm in terms of a special Hamiltonian set of universal quantum gates, namely

$$\hat{S}_{\pm X_j}(\Delta\phi) = e^{\mp i\hat{X}_j \Delta\phi}, \quad \hat{S}_{\pm Z_j}(\Delta\phi) = e^{\mp i\hat{Z}_j \Delta\phi},$$
$$\hat{S}_{\pm X_k X_j}(\Delta\phi) = e^{\mp i\hat{X}_k \hat{X}_j \Delta\phi}. \quad (5)$$

The Hamiltonians appearing in the exponents of equation (5) are themselves Pauli operators. Therefore, any unitary transformation $\hat{R}$ originating from Pauli operators either leaves these Hamiltonians invariant or changes their signs, such as

$$\hat{R}_j \hat{S}_{\pm X_j}(\Delta\phi)\hat{R}_j = \begin{cases} \hat{S}_{\pm X_j}(\Delta\phi) & \text{if } \hat{R}_j \in \{\mathbf{1}_j, \hat{X}_j\} \\ \hat{S}_{\mp X_j}(\Delta\phi) & \text{if } \hat{R}_j \in \{\hat{Y}_j, \hat{Z}_j\}. \end{cases} \quad (6)$$

Thus, with the help of these Hamiltonian quantum gates any change of the computational basis originating from a randomly selected set of Pauli operators can be compensated by an appropriate permutation of the universal quantum gates of equation (5).

On the basis of these considerations during a quantum computation an efficient correction of static errors can be achieved by the PAREC-method in the following way (compare with Fig. 1). In the first step randomly selected unitary operations from the set $\{\mathbf{1}, \hat{X}, \hat{Y}, \hat{Z}\}$ are applied to all n_q qubits of the QC, say $(\hat{X}_1, \hat{Y}_2, ..., \mathbf{1}_{n_q})$. The information about which qubit has been transformed by which Pauli operator is stored in a classical memory. In the second step one starts the quantum computation by applying a sequence of properly permuted universal quantum gates (compare with Eqs. (5) and (6)) as described by the first dashed box of Figure 1. This simple permutation does not require any extra significant computational effort. In the third step a second sequence of Pauli operators is selected randomly, say $(\hat{Z}_1, \hat{Y}_2, ..., \hat{X}_{n_q})$, and the combined quantum gates $(\hat{X}_1\hat{Z}_1, \hat{Y}_2\hat{Y}_2, ..., \mathbf{1}_{n_q}\hat{X}_{n_q})$ are applied to all
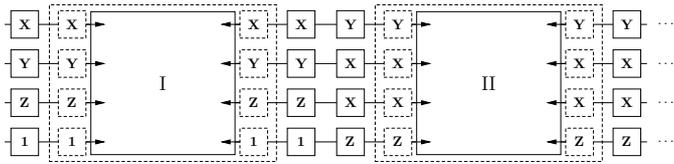
Fig. 1. The basic idea of the PAREC-method: the two boxes (full lines) represent two sequences of universal quantum gates for $n_q = 4$ qubits. Two random sequences of Pauli operators $(\hat{X}_1, \hat{Y}_2, \hat{Z}_3, \mathbf{1}_4)$ and $(\hat{Y}_1, \hat{X}_2, \hat{X}_3, \hat{Z}_4)$ are also indicated. The unitary Pauli operators outside the dashed boxes (full lines) are applied to the qubits whereas the ones inside the dashed boxes (dashed lines) are taken into account by appropriate permutations of the elementary quantum gates. Due to the identities $\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \mathbf{1}$ the inserted random sequences of Pauli operators change the computational basis but leave the ideal quantum algorithm unchanged.

the n_q qubits of the QC. These combined quantum gates are again Pauli operators. The information about the spin operators of the second selection is again stored in a classical memory. Afterwards the second sequence of properly permuted universal quantum gates is performed (second dashed box of Fig. 1). In the subsequent stages of the PAREC-method these steps are repeated after sequences of universal quantum gates of appropriate lengths n_{gef}. The influence of the choice of n_{gef} on the error correction will be discussed later (compare with Fig. 3). Finally, after the application of the last quantum gate the last randomly selected sequence of Pauli operators is applied to all qubits. As apparent from Figure 1 this PAREC-method leaves the ideal quantum algorithm unchanged. However, the repeatedly applied random unitary transformations produced by the Pauli operators change the signs of the parameters δ_i and J_i of equation (3) thus causing a randomization of the static errors. As a result we expect a significant improvement of the fidelity decay.

The stabilizing properties of the PAREC-method are investigated numerically in Figures 2 and 3 where it is applied to the iterative quantum tent-map. For this purpose the unitary operator of equation (1) is decomposed into a sequence of $\tilde{n}_g$ universal quantum gates of the form of equation (5). This can be achieved in a straightforward way, e.g., by corresponding replacements of the already known gate decomposition into n_g gates [12]. The PAREC-randomization is applied after appropriately chosen sequences of n_{gef} quantum gates of this latter decomposition of n_g gates. The influence of static imperfections is modeled by assuming that these universal quantum gates are performed instantaneously but that there is a certain time delay between any two successive quantum gates during which static imperfections cause errors. This time delay Δt models in an approximate way the time required for readjustments of the control unit of the QC before it can control the next quantum gate. As the quantum gates of equation (5) involve different accumulated phases $\Delta\phi$, within our model we also assume that quantum gates with larger phases require longer readjustment-times. Correspondingly, the readjustment-times Δt_1 and
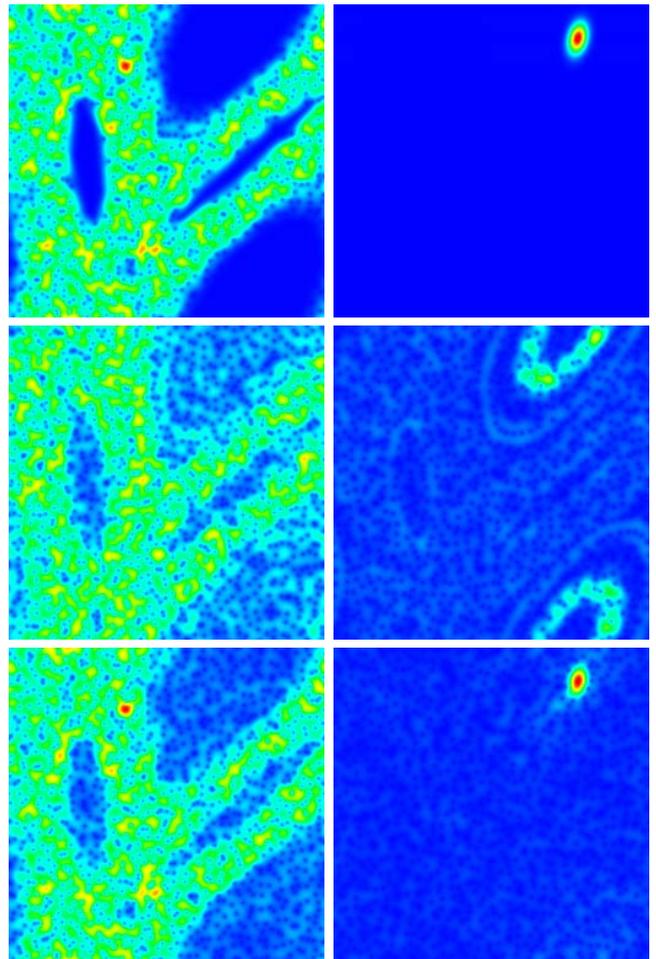
Fig. 2. Quantum Poincaré sections with Husimi-functions at $t = 3000$ in scaled momentum and position variables $\tilde{y} = p[(l/(2\pi\hbar)] \in [0, 2\pi]$ and $\tilde{x} = x[2\pi/l] \in [0, 2\pi]$: the parameters are $K = 1.7$ and $n_q = 10$. The initially prepared coherent states are centered around $(\pi/4, 0)$ (left panel) and $(5.35, 0)$ (right panel). First row: ideal dynamics; second row: static imperfections with $\epsilon = 5 \times 10^{-6}$; third row: PAREC-method applied after each sequence of $n_{\mathrm{gef}} = 20$ universal quantum gates of reference [12]. The probability density is coded in colors (red/maximum, blue/zero).
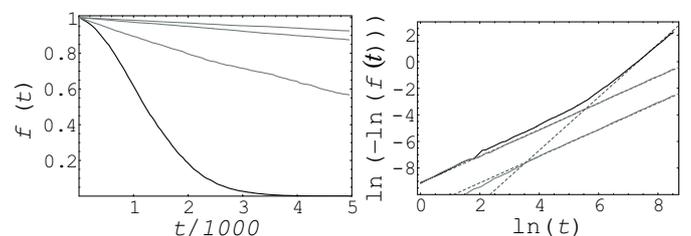
Fig. 3. Dependence of the fidelity $f(t)$ on the number of iterations: parameters as in the left panel of Figure 2; left: static imperfections without error correction, PAREC after each map iteration, after each $n_{\mathrm{gef}} = 50$, and after each $n_{\mathrm{gef}} = 20$ quantum gates (from bottom up); right: static imperfections without error correction, with PAREC after each map iteration and after each sequence of $n_{\mathrm{gef}} = 20$ quantum gates (full curves), best fits for linear- and quadratic-in-time decays (dashed curves).

Δt_2 after two successive quantum gates with accumulated phases $\Delta\phi_1$ and $\Delta\phi_2$ are related by $\Delta t_2/\Delta t_1 = \Delta\phi_2/\Delta\phi_1$. Thus, for a typical sequence of two quantum gates, e.g., the influence of static imperfections is modeled by

$$\cdots [e^{-i(\hat{H}\Delta t/\hbar)(\Delta\phi_{n+1}/\pi)}S_{X_iX_j}(\Delta\phi_{n+1})]$$
$$[e^{-i(\hat{H}\Delta t/\hbar)(\Delta\phi_n/\pi)}S_{X_k}(\Delta\phi_n)] \quad \cdots \quad (7)$$

with Δt denoting the readjustment-time associated with a phase change $|\Delta\phi| = \pi$. Correspondingly, the parameters characterizing the average strength of the influence of the static imperfections between successive quantum gates are $(\delta_i \Delta t/\hbar)$ and $(J_i \Delta t/\hbar)$ which are selected randomly and uniformly from the interval $[-\sqrt{3}\epsilon, \sqrt{3}\epsilon]$ and which remain constant during the quantum computation.

The ideal dynamics of the quantum tent-map is illustrated in the first row of Figure 2 where its Husimi-functions [13] are plotted after $t = 3000$ iterations for two initially prepared coherent states located close to the classically unstable (left) and to the classically stable (right) fixed points. In the left figure of the first row the initially prepared coherent state spreads almost uniformly over the classically chaotic component of phase space. Classically inaccessible regions originating from Kolmogorov-Arnold-Moser-tori are also apparent. The quantum probability leaking into these regions by quantum tunneling is still negligibly small. In the right figure of the first row the quantum probability is still concentrated in the region of the initially prepared coherent state. This reflects the approximately regular classical dynamics in this part of phase space. Static imperfections modify these dynamical characteristics significantly, as is apparent from the second row of Figure 2. Most prominently the influence of quantum tunneling into the classically inaccessible regions of phase space is no longer negligibly small. Furthermore, the detailed structures in both the regular and the chaotic parts of phase space are modified significantly. The corresponding results of the PAREC-method are depicted in the third row. Despite the fact that random sequences of Pauli operators are applied only after each sequence of $n_{\mathrm{gef}} = 20$ gates of reference [12] the corrected quantum states resemble the ideal case very closely.

The quantitative dependence of the fidelity on the number of iterations of the quantum tent-map is depicted in Figure 3 for the same parameters as in the left panel of Figure 2. The initially prepared coherent state of the right panel of Figure 2 yields similar results. It is apparent that the PAREC-method changes the fidelity decay from quadratic- to linear-in-time, i.e. $\ln f(t) = -t/t_c$. Thus, the time over which quantum computations can be performed reliably is increased significantly. In addition, one also notices that with increasing numbers of random unitary operations the decay rate $1/t_c$ also decreases. In particular, the best fits to the PAREC-results of Figure 3 together with further numerical studies suggest a dependence of the form $1/t_c = a\epsilon^2 n_q n_g n_{\mathrm{gef}}$ as long as $1 \ll n_{\mathrm{gef}} \ll n_g$. Thereby, n_{gef} is the effective number of original gates of reference [12] over which the influence of static imperfections adds up coherently. Our numerical data of Figure 3

give $a \simeq 1$. Physically speaking this dependence is plausible on the basis of the following heuristic consideration. If the PAREC-method is repeated after each map iteration, one observes $n_{\mathrm{gef}} = n_g$ as coherence is destroyed by the random basis changes and by the chaotic dynamics after each iteration. This result is consistent with numerical studies [12]. In the extreme opposite case in which the PAREC-method is repeated already after each universal quantum gate one expects $n_{\mathrm{gef}} = 1$ as coherence is destroyed by random basis changes already after each gate operation. The above mentioned dependence interpolates linearly between these two extreme cases. However, due to fluctuations it is expected that these considerations only apply for sufficiently large values of n_{gef}.

In summary, a general method for the correction of unitary static inter-qubit errors has been presented. This PAREC-method is particularly well suited to stabilize many-qubit systems against the disastrous effects of static imperfections in arbitrary quantum algorithms. In contrast to conventional quantum error correcting methods which exploit redundancy the PAREC-method does not require any extra qubits so that all physical qubits can be used in an optimal way.

References

1. M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge UP, 2000)
2. D. Deutsch, Proc. Roy. Soc. Lond. A **400**, 97 (1985)
3. P.W. Shor, In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, edited by S. Goldwasser (IEEE Computer Society, Los Alamitos, CA, 1994), p. 124
4. L.K. Grover, Phys. Rev. Lett. **79**, 325 (1997)
5. S. Lloyd, Science **273**, 1073 (1996)
6. R. Schack, Phys. Rev. A **57**, 1634 (1998); B. Georgeot, D.L. Shepelyansky, Phys. Rev. Lett. **86**, 2890 (2001)
7. W.H. Zurek, Rev. Mod. Phys. **75**, 715 (2003)
8. For a comprehensive list of pioneering work on quantum error correction see, e.g., reference [1]
9. For some more recent developments see, e.g., G. Alber, Th. Beth, Ch. Charnes, A. Delgado, M. Grassl, M. Mussinger, Phys. Rev. Lett. **86**, 4402 (2001); Ch. Ahn, H.M. Wiseman, G.J. Milburn, Phys. Rev. A **67**, 052310 (2003); A.M. Steane, preprint `arXiv:quant-ph/0304016`; E. Knill, preprint `arXiv:quant-ph/0404104`
10. B. Georgeot, D.L. Shepelyansky, Phys. Rev. E **62**, 3504 (2000); B. Georgeot, D.L. Shepelyansky, Phys. Rev. E **62**, 6366 (2000)
11. G. Benenti, G. Casati, S. Montangero, D.L. Shepelyansky, Phys. Rev. Lett. **87**, 227901 (2001)
12. K.M. Frahm, R. Fleckinger, D.L. Shepelyansky, Eur. Phys. J. D **29**, 139 (2004)
13. S.-J. Chang, K.-J. Shi, Phys. Rev. A **34**, 7 (1986)
14. F.M. Izrailev, Phys. Rep. **196**, 299 (1990)